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On the existence of multiple Kármán vortex-street modes

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An inviscid Kármán-type vortex-shedding model is employed to show that multiple modes of vortex streets are possible, for flow around a given obstacle. This is in confirmation of various experimental observations in recent years, which were challenged by opposing claims that these were due to experimental inaccuracies.

1. Introduction

Several experimental observations of periodic vortex trails behind stationary bodies in two-dimensional flow (Tritton 1959, 1971; Berger 1964) suggest that there are two different possible shedding modes for a certain range of Reynolds numbers. These modes are represented by two vortex streets of different streamwise and lateral spacing appearing alternately under the same experimental conditions, and two separate relationships between the frequency and oncoming flow speed (Reynolds number).

The question of the actual existence of these modes is still rather controversial (Berger & Wille 1972) owing to the work of Gaster (1969, 1971), who concluded that the apparently different modes resulted from non-uniform incoming flow on the obstacle producing the vortex street.

Taneda (1959), who had a unique experimental arrangement permitting observations very far downstream, presented evidence of rather long-lived stable vortex streets suddenly and spontaneously breaking down, and being re-established with larger streamwise and lateral spacing a short distance later. Experiments with decelerated flow (Durgin & Karlsson 1971) and multiple trails (Zdravkovich 1968) also show such behaviour.

This variety of data raises the possibility of alternative vortex–wake developments for a given combination of free-stream flow and obstacle. No theoretical treatment of this phenomenon has been found in the literature, and the purpose of this brief note is to test the possibility of such alternative wake developments occurring, using an inviscid Kármán-type model. The inviscid vortex-shedding approach has given surprisingly good results for different problems, from Kármán's original analysis of shedding from a cylinder to the recent work by Clements (1973) on bluff bodies.

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2. Analysis

The streamwise and lateral spacing of vortices shed from a stationary or moving body change in the vicinity of the obstacle (Weihs 1972), settling down (stability considerations aside) to constant values far downstream. In the above-mentioned paper (referred to as I from now on) the author examined thrust-type vortex trails and found that each set of data such as free stream velocity and body dimensions and movement results in a unique downstream trail, when requiring constant spacing far away from the obstacle.

In the present note, the drag-type vortex trails shed from stationary two-dimensional bodies in inviscid flow are studied.

Define the initial streamwise spacing $2d$ of the vortices as the distance between vortices due only to the oncoming velocity and not including the effects of induced velocity. The distances measured between successive vortices of the same sign and between rows far from the body are $2a$ and $2h$ respectively and are different from those measured closer to the obstacle because of effects of varying induced velocity on the vortices as they move downstream (I).

Assuming now that at a large distance from the body the vortex street becomes of regular shape, i.e. h and a are constants, one can find relations between the initial and final spacings. This assumption implies that the trails are stable. While it has been analytically shown that all inviscid Kármán-type streets are ultimately unstable (Kochin, Kibel & Roze 1964) experimental evidence shows the existence of regular trails for large distances (Taneda 1959).

By definition
$$2d = UT, \tag{1}$$
where U is the oncoming velocity and T the shedding period, and
$$2a = 2d - VT, \tag{2}$$
where V is the induced velocity of each vortex in a Kármán infinite vortex trail, i.e. (Milne-Thomson 1968)
$$V = \frac{k\pi}{2a}\tanh\frac{\pi h}{a}. \tag{3}$$

Here k is the absolute vortex strength. The fluid is taken to be inviscid, so that all vortices are of the same strength. Equation (3) may be used as a direct result of the requirement of constant vortex spacing far downstream, for when one gets far enough from the original disturbance the situation approaches that of the Kármán street stretching to infinity in both directions (I).

Substituting (1) and (3) into (2) we have
$$a = \tfrac{1}{2}UT\left(1 - \frac{k}{aU}\frac{\pi}{2}\tanh\frac{\pi h}{a}\right). \tag{4}$$

We now take the spacing ratio h/a to be a separate parameter and obtain a quadratic equation for a in terms of U, T, k and $\epsilon\ (\equiv h/a)$. Rearranging and solving this equation gives
$$a = \tfrac{1}{4}UT[1 \pm (1-\lambda)^{\frac{1}{2}}], \tag{5}$$
where
$$\lambda = 4\pi\sigma\tanh\pi\epsilon; \quad \sigma = k/U^2T.$$

λ is non-negative because each of the quantities included is separately positive. As a result, there are two distinct physically possible (positive) values of the streamwise spacing a for each value of the parameters when $0 < \lambda < 1$, a single solution when $\lambda = 0$ or $\lambda = 1$ and no real solutions when $\lambda > 1$. This result is significant in two ways. First, it defines a range of the parameter λ for which periodic shedding is obtained. Then it shows that, within this range, there are always (except for two limiting cases) two possible vortex streets with the same spacing ratio for any body placed in a given stream.

Looking at the limiting cases, we see that $\lambda = 0$ is fulfilled only when h or k vanishes, or there is no periodic shedding. In other words, this is a trivial case of no vortex street, or of zero drag ($h = 0$).

The previous analysis has shown that, starting from the classical case described by Kármán, one can obtain two different vortex streets, with equal spacing ratios but different streamwise and lateral spacing. This basic result can be carried somewhat further by taking the drag force into account.

The mean drag force on the obstacle producing the vortex streets is, in the present notation (Kochin *et al.* 1964),

$$D = \frac{\rho\pi k^2}{a} + \frac{\rho\pi kh}{a}(U-2V). \tag{6}$$

Substituting the two possible values of a from (5) separately into (6) one obtains two distinct values for the drag on the body. In the cases reported by Berger (1964) and Tritton (1959, 1971) this result is physically plausible, for in these experiments the switching from mode to mode occurred near the cylinder. The presence of a different configuration of vortices would lead to changes in the induced velocity field at the body. In the case of a circular cylinder this would result in changes in the base pressure and in the drag† experienced. Some experimental indication of this appears in Tritton's (1959) experiments on the deflexions of fibres shedding vortex streets.

On the other hand, the spontaneous change from one street to the other far downstream, as in the experiments reported by Taneda (1959), would have negligible effects upon the drag. In this situation the analysis in the previous section must be modified. Here the argument is that the drag on the obstacle is a constant. As a result some other constraint must be relaxed so that we do not have an over-defined system.

In such long-lived vortex streets, viscosity, however small, has time to act upon the vortices, so that the former assumption of constant vortex strength k is less justified in this case. We shall therefore now allow k to vary. However, we still use (3) and (6), which were derived for vortex streets with constant k. This implies that each vortex street has constant k, but that these vortex strengths are not equal. In terms of the observed transition from one trail to the other, we now assume that all dissipation of vorticity takes place during transition.

We have now the three equations, (2), (3) and (6), for the variables a, k and V, which are functions of parameters D, ϵ, U and T. From (2) and (6),

$$k^2(1-\pi\epsilon\tanh\pi\epsilon)/a + k\epsilon U - D' = 0, \tag{7}$$

† The author is grateful to Prof. R. Wille for pointing this out to him.

where $D' = D/\rho\pi$. Also from (2) and (3)

$$k = (U - 2a/T)(2a/\pi \tanh \pi\epsilon). \tag{8}$$

After substitution of (8) into (7) and some further manipulations,

$$a^3 + \beta_2 a^2 + \beta_1 a + \beta_0 = 0 \quad (z \neq 1), \tag{9}$$

where

$$\beta_2 = -UT\left[1 + \frac{z}{2(1-z)}\right], \quad \beta_1 = \frac{(UT)^2}{4}\left[1 + \frac{z}{4(1-z)}\right], \quad \beta_0 = -\frac{D'T^2z^2}{16\epsilon^2(1-z)}$$

and

$$z = \pi\epsilon \tanh \pi\epsilon. \tag{9a}$$

When $z = 1$ ($\epsilon \simeq 0{\cdot}382$), a quadratic equation for a is obtained, as is obvious from (7):

$$a^2 - \tfrac{1}{2}aUT + (D'Tz/4U\epsilon^2) = 0. \tag{10}$$

The solutions of (10) are, recalling that $z = 1$ here,

$$a = \tfrac{1}{4}UT[1 \pm (1 - 4D'/\epsilon^2U^3T)^{\frac{1}{2}}], \tag{11}$$

i.e. there again are two distinct positive values of a. Returning to (9) and using the criteria for the number of real solutions (Abramowitz & Stegun 1965), it can be shown that for all values of the spacing ratio $0 < \epsilon < 1$ except those close to $\epsilon = 0{\cdot}382$ there are three distinct real solutions. In establishing this general result there is some difficulty as the coefficient β_0 includes the drag, while the others include only the initial longitudinal spacing UT. The expression independent of ϵ in β_0 is $D'T^2$. This can be written as

$$D'T^2 = DT^2/\rho\pi = \tfrac{1}{2}\rho U^2 A_w C_D T^2/\rho\pi, \tag{12}$$

where A_w stands for wetted area. In the case of a two-dimensional circular cylinder $A_w = 2\pi r$ per unit length, where r is the radius. The Strouhal number $S \equiv 2r/UT$, and so finally

$$D'T^2 = \tfrac{1}{2}SC_D(UT)^3. \tag{13}$$

The drag coefficient and Strouhal numbers for vortex shedding from cylinders are known empirically and a relatively accurate estimate of the factor SC_D can be made, being of order 0·1.

Having found that in the region of interest ($\epsilon \simeq 0{\cdot}3$) there are three real solutions to (9), it now remains to determine their sign. For $\epsilon < 0{\cdot}382$, $1-z$ is positive, so that $\beta_2, \beta_0 < 0$ and $\beta_1 > 0$ in this range. From the general properties of the cubic equation we have (Abramowitz & Stegun 1965)

$$\prod_{i=1}^{3} a_i = -\beta_0. \tag{14}$$

From (14) we see that either one, or all three roots of (9) must be positive. If only one root is positive, then its absolute value (call it a_1) must be larger than the sum of the absolute values of the two other roots, because

$$\sum_{i=1}^{3} a_i = -\beta_2. \tag{15}$$

But the cubic equation also fulfils

$$a_1 a_2 + a_1 a_3 + a_2 a_3 = \beta_1. \tag{16}$$

β_1 is positive in (9), but if only $a_1 > 0$ then, as a result of the former condition (14), condition (16) implies that $\beta_1 < 0$, as $|a_1 a_2| > |a_2 a_3|$ and $|a_1 a_3| > |a_2 a_3|$. Therefore, the case of one positive root only is ruled out, and we obtain the result that in the region of interest (9) has three distinct possible positive solutions.

3. Discussion

The interesting and rather unexpected result of the analysis above, that the Kármán inviscid vortex-street model allows for multiple trail modes, strengthens the case of the experimenters who obtained such modes. The obvious next step to establish beyond doubt the existence of alternative modes is to correlate (5) and/or (9) with experimental data.†

The assumption of equal spacing ratios for the different modes, which is the only additional one required in the analysis of the previous section, is not essential. Taking h/a to be variable between the possible modes leads to equations which give multiple solutions for a dependent upon the function $h = h(a)$. These were obtained, for (4), by expanding $h(a)$ as a Taylor series.

The main difficulty here is that the actual form of this function is not known. As a result, the present analysis used only h/a = constant for a given combination of obstacle and velocity. This is partially justified from inviscid stability theory, which shows that certain spacing ratios are preferred. Also, examination of Taneda's and Berger's flow visualizations tends to confirm that the spacing ratios of the 'primary' and 'secondary' streets are approximately equal. This is a very crude approximation taken from the published data, but more accurate measurements of lateral spacings for the two modes have not been found.

From (5), we have for the ratio of streamwise spacings

$$\frac{a_2}{a_1} = \frac{1+(1-\lambda)^{\frac{1}{2}}}{1-(1-\lambda)^{\frac{1}{2}}}. \tag{17}$$

Taneda (1959) has tabulated a number of experimental values for the ratio a_2/a_1 (in his table II). It must be recalled that these measurements were made

† A referee has pointed out in this connexion that, while actual lateral and streamwise vortex-street spacings are both observed to increase as one moves downstream, (5) suggests the opposite. This result is an artifact of the inviscid Kármán vortex-street model, as vortices near the obstacle have only a semi-infinite street (I) inducing a retarding velocity V, while vortices far downstream 'see' an infinite street, stretching in both directions. The referee also suggested a simple way of removing this difficulty: modifying the definition of initial spacing to include the mechanics of formation of the discrete vortices by putting, instead of (1),

$$2d = \alpha U T, \tag{1a}$$

where $\alpha \leqslant 1$ is an empirically found constant. Repeating the analysis gives

$$\frac{a}{d} = \frac{1}{2\alpha}[1 \pm (1-\lambda)^{\frac{1}{2}}], \tag{5a}$$

so that $a > d$ as in experiment, while the main conclusion on the existence of two distinct vortex-street modes is unchanged. A similar modification can be carried out on (9).

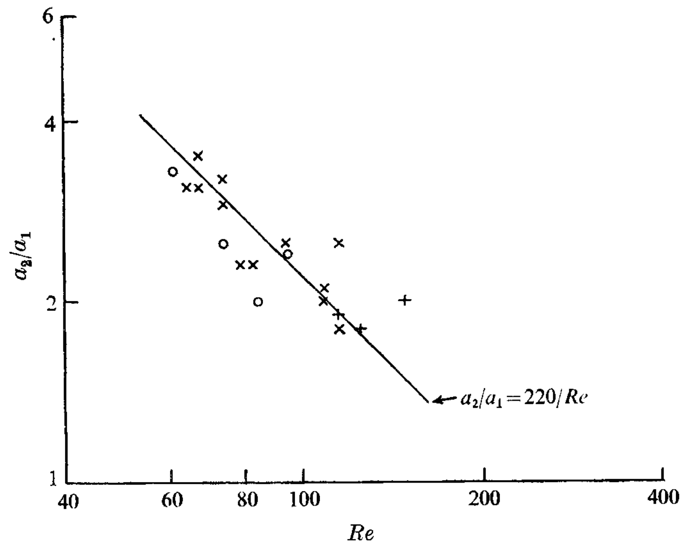

FIGURE 1. Ratio of streamwise spacing of secondary to primary vortex streets downstream of a circular cylinder, measured by Taneda (1959), *vs.* Reynolds number. Cylinder diameter: ○, 0·2 cm; ×, 0·3 cm; +, 0·48 cm.

with aluminium particles in a large towing tank. The inferred breakdown and re-establishment of vortex streets may therefore be due to unobserved influences such as residual currents and uneven reflexions in this case, and not describe multiple modes at all. Assuming, however, that this is a genuine case of such modes appearing, his results have been plotted (see figure 1) as a function of the Reynolds number, and an approximate empirical relationship was found. From figure 1,

$$a_2/a_1 \simeq A/Re, \tag{18}$$

where $A \simeq 220$. Using (18)

$$\lambda = 1-[(A-Re)/(A+Re)]^2, \tag{19}$$

which shows the rather weak dependence of λ on the Reynolds number. It varies from $\lambda = 0{\cdot}77$ when $Re = 60$ to $\lambda = 0{\cdot}91$ at $Re = 120$, which is the range of Reynolds numbers for which multiple modes are observed. Consequently any correlations attempted with experimental data have to be based on very accurate measurements.

Attempts to obtain a general comparison by means of empirical relationships between the constituents of a_i have not been very significant. The dependence of the vortex strength and the spacing ratio upon the Reynolds number or external dimensional variables is not known precisely enough. Trial computations were made on the basis of relationships such as

$$k = \Lambda U^2 T, \quad SRe = A\,Re - B, \tag{20}$$

where Λ, A and B are empirical constants and S the Strouhal number (Berger 1964). These gave results of the right order of magnitude but this agreement has to be treated as merely fortuitous for the present. It is clear therefore that more experimental data, with special emphasis on the unsteady drag, spacing ratios and degree of absorption of shed vorticity in specific situations of multiple-mode shedding, are required.

These would firmly establish whether the present result of multiple modes of shedding, obtained by the greatly idealized Kármán model, is a true description of the rather controversial dual-mode shedding data.

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